Thursday, April 16

Term projects presentation schdule

	April 23	April 28	April 30
9:00	Tomas Amadeo	Shifan Wei	Svetlana Bakhmatova
9:15	Jingci Li	Joanna Thelen	Hanqing Liu
9:30	Qingyang Long	Lanqi Li	Lijie Liu
9:45		Sriram Ravichandran	Haochen Li
10:00	Zekun Wang	Feiyang Kang	Linqianhao Liu
10:15	Qirong Na	Evelyn Liu	Lingyan Jiang
10:30		Namrah Khan	Sanika Barve

Whittaker, E.T., & McCrae, S. (1988). *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge Mathematical Library). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511608797

Principle of Least-Action

The starting point is the *action*, denoted $\mathcal{S}$ of a physical system. It is defined as the integral of the Lagrangian L between two instants of time t_1 and t_2 – technically a functional of the N generalized coordinates $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ which define the configuration of the system:

$$\mathbf{q} : \mathbf{R} \to \mathbf{R}^N$$

$$\mathcal{S}[\mathbf{q}, t_1, t_2] = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt$$

where the dot denotes the time derivative, and t is time.

Mathematically the principle

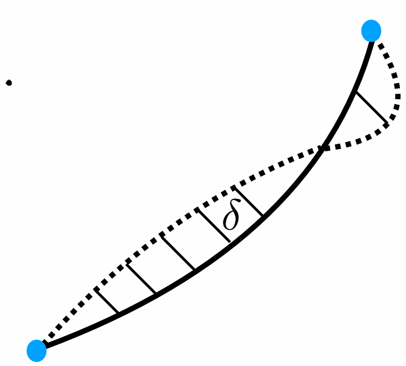
$$\delta \mathcal{S} = 0,$$

where δ (lowercase Greek delta) means a *small* change. In words this reads:

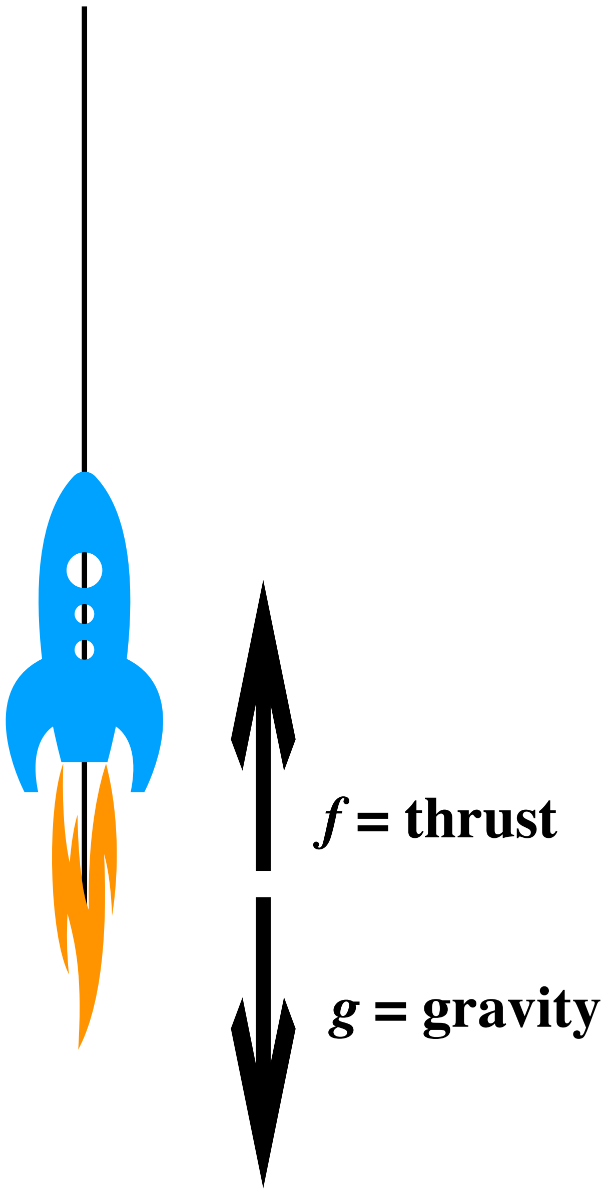
*The path taken by the system between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the **action** is **stationary (no change)** to **first order**.*

In applications the statement and definition of action are taken together:

$$\delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt = 0.$$



$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}}$$



$$f - \mathfrak{m}g = \mathfrak{m}\ddot{x}. \tag{8.4}$$

We now apply the Lagrangian formalism to derive the same result. The kinetic energy is $\mathfrak{m}\dot{x}^2/2$, the potential energy is $\mathfrak{m}gx$, and the Lagrangian is

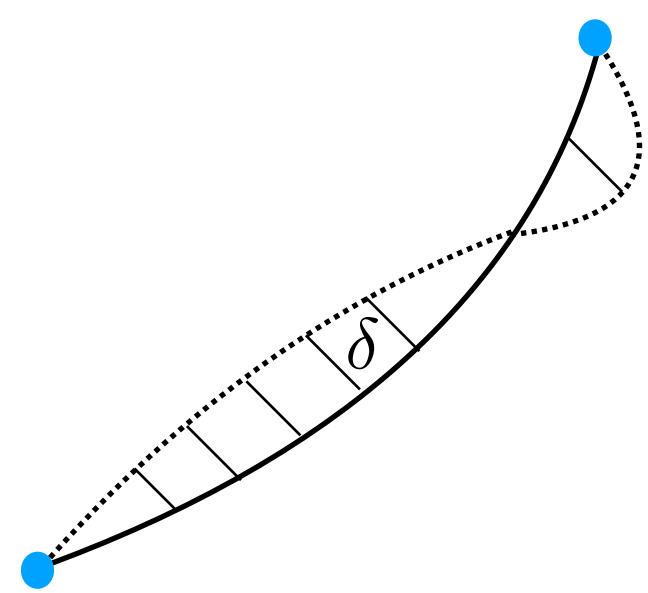
$$\mathcal{L}(x, \dot{x}) = \mathcal{K}(x, \dot{x}) - \mathcal{P}(x) = \frac{1}{2}\mathfrak{m}\dot{x}^2 - \mathfrak{m}gx. \tag{8.5}$$

The equation of motion is then given by

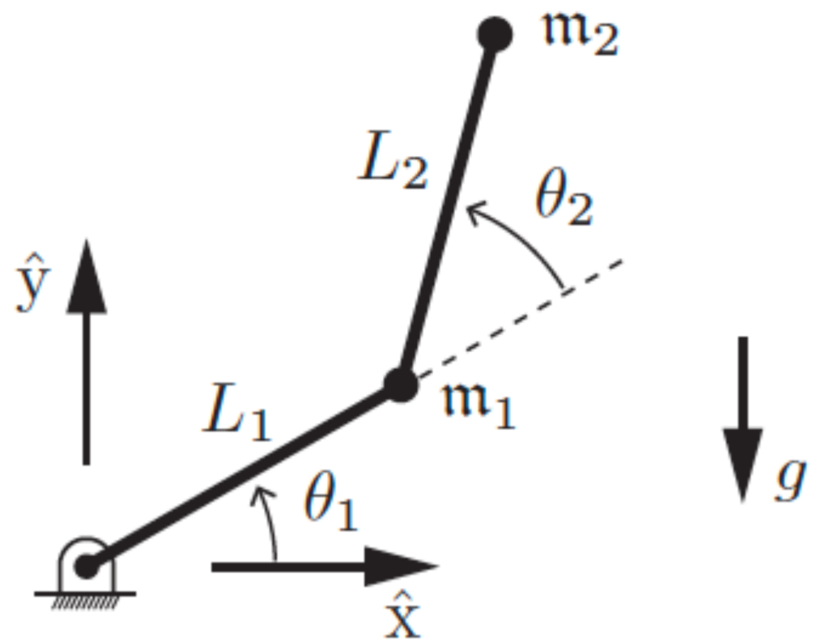
$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \mathfrak{m}\ddot{x} + \mathfrak{m}g, \tag{8.6}$$

which matches Equation (8.4).

— Lynch and Park



Lagrangian mechanics of kinematic chains



Fully actuated mechanical systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}_{\mathbf{i}}} - \frac{\partial L}{\partial \mathbf{q}_{\mathbf{i}}} = u_i$$

$$\Downarrow$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = u_i$$

the ends of each link. The position and velocity of the link-1 mass are then given by

$$\begin{aligned}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix}, \\
\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} &= \begin{bmatrix} -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \end{bmatrix} \dot{\theta}_1,
\end{aligned}$$

while those of the link-2 mass are given by

$$\begin{aligned}
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) \end{bmatrix}, \\
\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2) & -L_2 \sin(\theta_1+\theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) & L_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.
\end{aligned}$$

We choose the joint coordinates $\theta = (\theta_1, \theta_2)$ as the generalized coordinates. The generalized forces $\tau = (\tau_1, \tau_2)$ then correspond to joint torques (since $\tau^{\mathrm{T}}\dot{\theta}$ corresponds to power). The Lagrangian $\mathcal{L}(\theta, \dot{\theta})$ is of the form

$$\mathcal{L}(\theta, \dot{\theta}) = \sum_{i=1}^{2} (\mathcal{K}_i - \mathcal{P}_i), \tag{8.7}$$

where the link kinetic energy terms $\mathcal{K}_1$ and $\mathcal{K}_2$ are

$$\begin{aligned}
\mathcal{K}_1 &= \frac{1}{2}\mathfrak{m}_1(\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2}\mathfrak{m}_1 L_1^2 \dot{\theta}_1^2 \\
\mathcal{K}_2 &= \frac{1}{2}\mathfrak{m}_2(\dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}\mathfrak{m}_2 \left((L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\dot{\theta}_1^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2 + L_2^2\dot{\theta}_2^2 \right),
\end{aligned}$$

and the link potential energy terms $\mathcal{P}_1$ and $\mathcal{P}_2$ are

$$\begin{aligned}
\mathcal{P}_1 &= \mathfrak{m}_1 g y_1 = \mathfrak{m}_1 g L_1 \sin\theta_1, \\
\mathcal{P}_2 &= \mathfrak{m}_2 g y_2 = \mathfrak{m}_2 g (L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2)).
\end{aligned}$$

Underactuated (super-articulated) mechanical systems

The pendubot

—D.J. Bloch & M. Spong

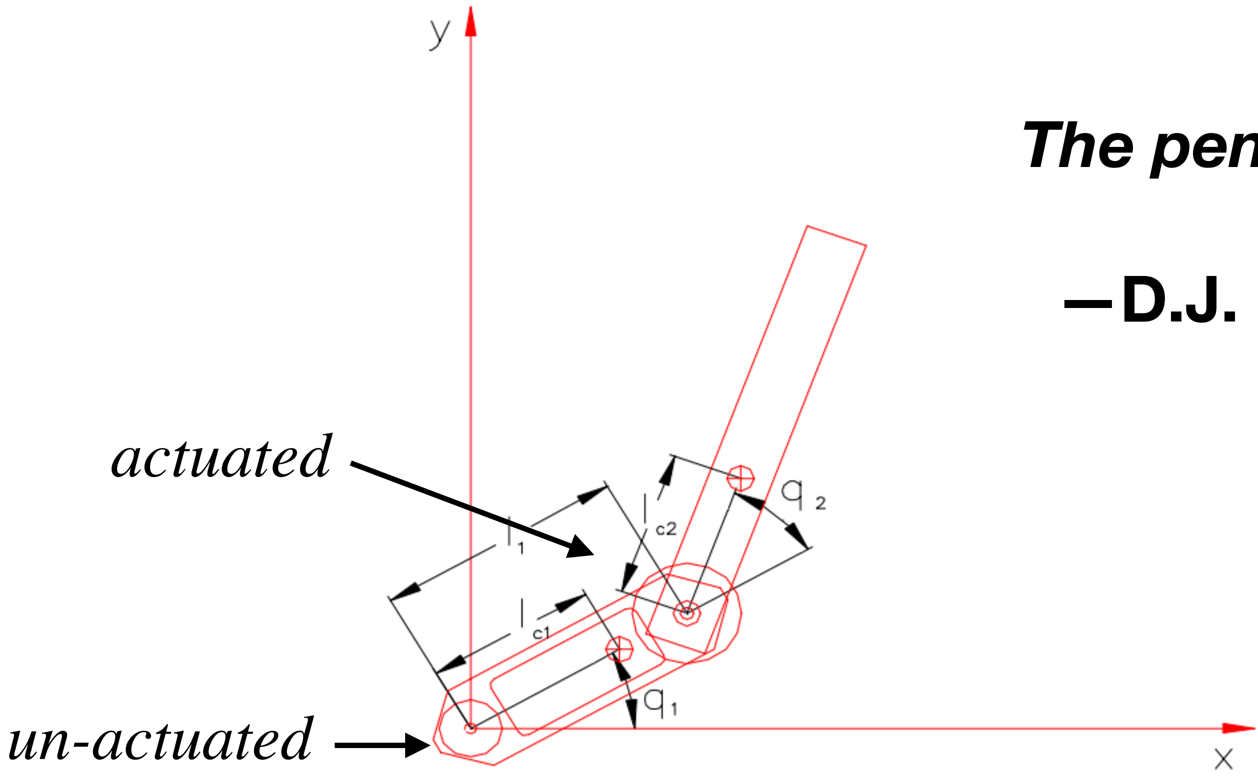


Figure 3.1 Coordinate Description of the Pendubot. l_1 is the length of link one, l_{c1} and l_{c2} are the distances to the center of mass of the respective links and q_1 and q_2 are the joint angles of the respective links.

The equations of motion for the Pendubot can be found using Lagrangian dynamics [5]. In matrix form the equations are

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \qquad (3.1)$$

where τ is the vector of torque applied to the links and $\mathbf{q}$ is the vector of joint angle positions with

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \qquad \begin{aligned} d_{11} &= m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos q_2) + I_1 + I_2 \\ d_{12} &= d_{21} = m_2(l_{c2}^2 + l_1 l_{c2} \cos q_2) + I_2 \\ d_{22} &= m_2 l_{c2}^2 + I_2 \end{aligned} \qquad (3.2)$$