# ME/SE 740

# Lecture 9

# Euler Angles and Euler's Theorem

**Euler Angles**
Today we return to our examination of rotations of rigid bodies. We begin with a discussion about "Euler Angles." Consider the two rotations depicted in the figure below where the "north pole" is first rotated about the z-axis through an angle $\theta$ and this is followed by a rotation about the y-axis through an angle $\phi$. The north pole ends up at point $Q$ on the unit sphere.

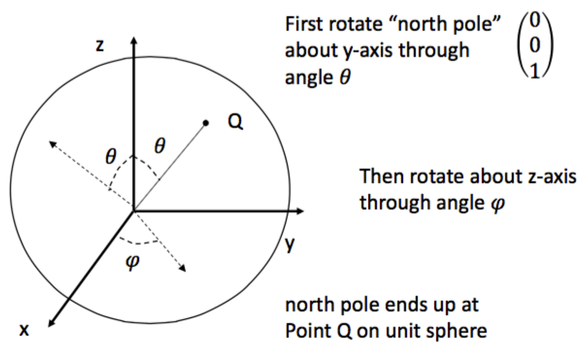


Figure 1: **Two Rotations of the North Pole**

These two rotations can be represented by:

$$\begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Suppose we are given an arbitrary $3 \times 3$ rotation (of the north pole):

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and that both map the north pole $(0, 0, 1)^T$ to the same point:

$$\begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Multiplying on the left both sides of this equation by the matrix inverses (in the appropriate order) we can write:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}}_{T} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where $T$ maps $(0,0,1)^T$ to $(0,0,1)^T$ .

Given a left action of a group $G : M \longrightarrow M$ on a manifold $M$ and a point $m \in M$ the set of elements $g \in G$ such that $g \circ m = m$ is a subgroup called the isotropy subgroup of $M$. We say that $G$ has a left action on $M$ if for every $g \in G, g_1 \circ (g_1 \circ m) = (g_1 \circ g_2)m$ for $g_1, g_2 \in G$ and $m \in M$. In particular, the isotropy subgroup of $SO3)$ ( $3 \times 3$ proper rotation matrices) corresponding to $(0,0,1)^T$ is:

$$\begin{pmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Psi$$

There is a value of $\psi$, with $0 \leq \psi 2\pi$ such that

$$\begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Solving for $\vec{n}, \vec{o}, \vec{a}$ we obtain:

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \underbrace{\begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{0 \,\leq\, \phi \,<\, 2\pi} \underbrace{\begin{pmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{pmatrix}}_{0 \,\leq\, \theta \,\leq\, \pi} \underbrace{\begin{pmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{0 \,\leq\, \psi \,<\, 2\pi}$$

**Tait-Bryan Angles**

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \underbrace{\begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{0 \,\leq\, \phi \,<\, 2\pi} \underbrace{\begin{pmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{pmatrix}}_{0 \,\leq\, \theta \,\leq\, \pi} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{pmatrix}}_{0 \,\leq\, \psi \,<\, 2\pi}$$

**Euler's Theorem** (for us). Every $3 \times 3$ proper rotation $X \in SO(3)$ is a rotation about an axis by a certain amount $\theta, 0 \leq \theta \leq \pi$.

Suppose that $X$ is given:

$$X = \begin{pmatrix} \bar{n}_x & \bar{o}_x & \bar{a}_x \\ \bar{n}_y & \bar{o}_y & \bar{a}_y \\ \bar{n}_z & \bar{o}_z & \bar{a}_z \end{pmatrix}$$

By Euler's Theorem, there is a unit vector $\vec{k}$ and a rotation $0 \leq \theta \leq \pi$ such that $X$ is a rotation of $\theta$ about $\vec{k}$.

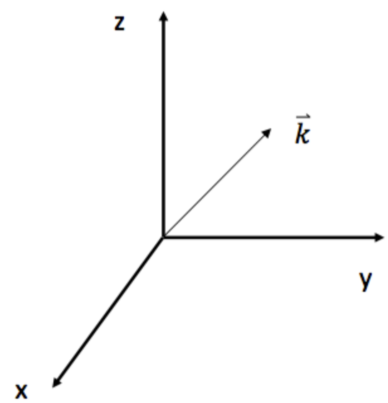


Figure 2: **The axis $\vec{k}$**

Let $C$ map $(0,0,1)^T$ to $\vec{k}$, $\vec{k} = C(0,0,1)^T$. This implies:

$$\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

There are two coordinate systems in which I want to consider rotation of $\theta$ radians about $\vec{k}$. Consider an arbitrary point on the sphere whose $C$ frame coordinates and the base frame coordinates are:

$$\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix}, \quad \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = C \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix}, \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = C^T \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

respectively before rotation. They are:

$$\begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} \text{ and } C \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix}$$

after rotation respectively. Hence in terms of Base coordinates:

$$\begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \longrightarrow C \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} C^T \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

Let

$$C = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}$$

The above product

$$\begin{aligned} C \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} C^T &= \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \\ &= \begin{pmatrix} n_x c\theta + o_x s\theta & -n_x s\theta + o_x c\theta & a_x \\ n_y c\theta + o_y s\theta & -n_y s\theta + o_y c\theta & a_y \\ n_z c\theta + o_z s\theta & -n_z s\theta + o_z c\theta & a_z \end{pmatrix} \begin{pmatrix} n_x & n_y & n_z \\ o_x & o_y & o_z \\ a_x & a_y & a_z \end{pmatrix} \end{aligned}$$

The elements of this matrix product can be expressed as:

$$\begin{aligned} \text{element}(1,1) &: n_x^2 c\theta + n_x o_x s\theta - n_x o_x s\theta + o_x^2 c\theta + a_x^2 && = (n_x^2 + 0_x^2)c\theta + a_x^2 \\ \text{element}(2,1) &: n_y n_x c\theta + n_x o_y s\theta - n_y o_x s\theta + o_x 0_y c\theta + a_x a_y && = (n_x n_y + o_x o_y)c\theta + (n_x o_y - n_y o_x)s\theta + a_x a_y \\ \text{element}(3,1) &: n_z n_x c\theta + n_x o_z s\theta - n_z o_x s\theta + o_z o_x c\theta + a_x a_z && = (n_z n_x + o_x o_z)c\theta + (n_x o_z - n_z o_x)s\theta + a_x a_z \\ \text{element}(1,2) &: (n_x n_y + o_x o_y)c\theta + (o_x n_y - n_x o_y)s\theta + a_x a_y \\ \text{element}(2,2) &: (n_y^2 + o_y^2)c\theta + a_y^2 \\ \text{element}(3,2) &: (n_z n_y + o_z o_y)c\theta + (n_y o_z - n_z o_y)s\theta + a_y a_z \end{aligned}$$

Recall that for $i = x, y, z$ we have $n_i^2 + o_i^2 + a_i^2 = 1$, and that $n_i n_j + o_i o_j + a_i a_j = 0 \ \text{ for } \ i \neq j$, $(CC^T = I)$.

Furthermore, $\vec{n} \times \vec{o} = \vec{a}$.

$$\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix} \begin{pmatrix} o_x \\ o_y \\ o_z \end{pmatrix}$$

The very complicated matrix above may be written more simply by eliminating $n_i's, o_i's$. In particular, the $(1,1)$ entry can be written as:

$$k_x^2 + (1 - k_x^2)c\theta = c\theta + k_x^2(1 - c\theta)$$

Continuing in this manner we can express every $3 \times 3$ rotation in the form:

$$\begin{pmatrix} k_x^2(1-\cos\theta) + \cos\theta & -k_z \sin\theta + k_x k_y(1-\cos\theta) & k_y \sin\theta + k_x k_z(1-\cos\theta) \\ k_z \sin\theta + k_x k_y(1-\cos\theta) & k_y^2(1-\cos\theta) + \cos\theta & -k_x \sin\theta + k_y k_z(1-cos\theta) \\ -k_y \sin\theta + k_x k_z(1-\cos\theta) & k_x \sin\theta + k_y k_z(1-\cos\theta) & k_z^2(1-\cos\theta) + \cos\theta \end{pmatrix}$$

If we are given:

$$\begin{pmatrix} \bar{n}_x & \bar{o}_x & \bar{a}_x \\ \bar{n}_y & \bar{o}_y & \bar{a}_y \\ \bar{n}_z & \bar{o}_z & \bar{a}_z \end{pmatrix}$$

we can solve for $(k_x, k_y, k_z)^T$ and $\theta$. Take the trace of both sides:

$$\begin{aligned}
\bar{n}_x + \bar{o}_y + \bar{a}_z &= k_x^2(1-\cos\theta) + \cos\theta + k_y^2(1-\cos\theta) + \cos\theta + k_z^2(1-\cos\theta) + \cos\theta \\
&= (1-\cos\theta) + 3\cos\theta \\
&= 1 + 2\cos\theta \\
\implies \cos\theta &= \frac{\bar{n}_x + \bar{o}_y + \bar{a}_z - 1}{2}
\end{aligned}$$

unique solution if $0 \leq \theta \leq \pi$. Solve for $\theta$. Then solve for $k_x, k_y, k_z$ by looking at the off-diagonal terms.

$$\begin{aligned}
k_z \sin\theta &= \frac{\bar{n}_y - \bar{o}_x}{2} \\
k_y \sin\theta &= \frac{\bar{a}_x - \bar{n}_z}{2} \\
k_x \sin\theta &= \frac{\bar{o}_z - \bar{a}_y}{2}
\end{aligned}$$