ME/SE 740

Lecture 8

Matrix Exponentials

Motivation for today's lecture: In robotics we frequently encounter the rotation of some rigid body about some given axis by some amount. Consider the rotation of some link about the z-axis as shown below:

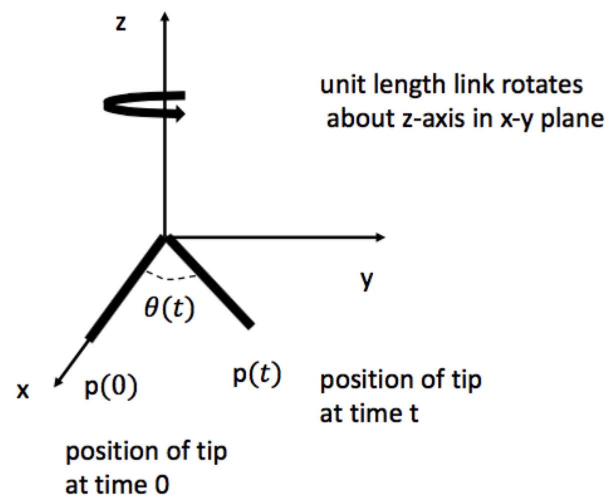

Figure 1: **Link Rotating**

Suppose $\theta(t)$ is the rotation angle in radians and suppose the link rotates at a constant, unit velocity, so that: $\theta(t) = t, \;\; \dot{\theta}(t) = 1$. As a result, the coordinate of the link tip at time $t = 0$ and $t = 1$ can be respectively expressed as:

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$$

The velocity of the tip is therefore given by:

$$\dot{p}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}$$

Which can be be expressed as:

$$\dot{p}(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A} p(t)$$

$$\dot{p}(t) = Ap(t)$$

This is a differential equation in state-space form with A being a constant matrix.

In general, if the axis of rotation is given by some unit vector:

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

one can show that:

$$\dot{p}(t) = \underbrace{\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}}_{A} p(t) = \underbrace{w \times p(t)}_{\text{cross product}}$$

Again a differential equation in state-space form:

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0$$

If we know $p(0)$ and w, we can compute $p(t)$ by solving this differential equation.

Theorem 0: Let A be an $n \times n$ matrix with constant entries and let the sequence of matrices be defined recursively as follows:

$$\begin{aligned} & M_0 = I \\ k \geq 1, \qquad & M_k(t,0) = I + \int_0^t AM_{k-1}(\sigma, 0)d\sigma \end{aligned}$$

The the sequence of matrices $M_0, M_1, M_2, \cdots$ converges uniformly on any time interval $0 \leq t \leq t_1$. Moreover, if the limit is defined as $\Phi(t,0)$ (i.e., $\Phi(t,0) = \lim_{k\to\infty} M_k(t,0)$) then for $0 \leq t \leq t_1$:

$$\frac{d\Phi(t,0)}{dt} = A\Phi(t,0), \qquad \Phi(0,0) = I$$

and the solution of $\dot{x}(t) = Ax(t), \quad x(0) = x_0$, is given by $x(t) = \Phi(t,0)x_0$.

Before we prove this theorem it will be helpful if we recall some basic facts/definitions/statements from the theory of convergence of sequences of functions.

D/F/S 1: Let $f_i(t)$ be a sequence of scalar real valued functions defined on the interval $T: 0 \leq t \leq t_1$. A sequence of functions $f_1(t), f_2(t), \cdots$ define on T, is said to <u>converge</u> (pointwise) to some function $f(t)$ if

$$f(t) \ = \ \lim_{n\to\infty} f_n(t) \quad for \ \ every \ \ t \in T$$

D/F/S 2: A sequence is said to converge uniformly to $f(t)$ on T, if for every $\epsilon > 0$, there exists an N (depending on ϵ not t) such that for $n > N$

$$|f_n(t) \ - \ f(t)| < \epsilon \ \ for \ \ every \ \ t \in T$$

D/F/S 3: A series of functions $f_1(t) + f_2(t) + f_3(t) + \cdots$ defined on T is said to converge to a function $f(t)$ if the sequence of partial sums $\{s_i(t)\}$ converges to $f(t)$ where:

$$\begin{aligned} s_1(t) &= f_1(t) \\ s_2(t) &= f_1(t) + f_2(t) \\ &\vdots \\ s_2(t) &= f_1(t) + f_2(t) + \cdots + f_n(t) \end{aligned}$$

D/F/S 4: The series $f_1(t)+f_2(t)+f_3(t)+\cdots$ converges uniformly to $f(t)$ on T, if the sequence $\{s_i(t)\}$ converges uniformly to $f(t)$ on T.

D/F/S 5: Theorem (Weirstrass M-test). Let $\{K_n\}$ be a sequence of non-negative numbers such that $0 \leq |f_n(t)| \leq K_n$ for $n = 1, 2, 3, \cdots$ and every $t \in T$. Then the series $f_1(t) + f_2(t) + f_3(t) + \cdots$ converges uniformly to $f(t)$ on T if $K_1 + K_2 + K_3 + \cdots$ converges.

D/F/S 6: Let $A, B, A_1, A_2, \cdots, A_k$ be $n \times n$ matrices with constant entries. Denote the i, j element of some matrix A as $E_{i,j}(A)$. Let $\alpha = \max_{i,j} |E_{i,j}(A)|, \quad \beta = \max_{i,j} |E_{i,j}(B)|, \quad \alpha_\ell = \max_{i,j} |E_{i,j}(A_\ell)|$. Then

$$\underbrace{|E_{i,j}(AB)|}_{i,j \ \ element \ of \ \ AB} \quad \leq n\alpha\beta$$

With $A = (a_{i,j}), \quad B = (b_{i,j})$ then:

$$\begin{aligned} \underbrace{|a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j}|}_{i,j \ \ element \ \ of \ \ AB} &\leq |a_{i,1}b_{1,j)}| + |a_{i,2}b_{2,j}| + \cdots + |a_{i,n}b_{n,j}| \\ &\leq |a_{i,1}||b_{1,j}| + |a_{i,2}||b_{2,j}| + \cdots + |a_{i,n}||b_{n,j}| \\ &\leq \alpha\beta + \alpha\beta + \cdots \alpha\beta \\ &\leq n\alpha\beta \end{aligned}$$

In general (via a proof by induction) one can show:

$$|E_{i,j}(A_1 A_2 \cdots A_k)| \leq n^{k-1} \alpha_1 \alpha_2 \cdots \alpha_k$$

proof of the Theorem

Step1: Obtain expressions for the sequence of matrices $M_0(t,0), M_1(t,0), M_2(t,0), \cdots$

$$
\begin{aligned}
M_0(t,0) &= I \\
M_1(t,0) &= I + \int_{\sigma=0}^{t} A d\sigma = I + A \int_{\sigma=0}^{t} 1 d\sigma \\
&= I + At \\
M_2(t,0) &= I + \int_{\sigma=0}^{t} A M_1(\sigma,0) d\sigma = I + \int_{\sigma=0}^{t} A(I + A\sigma) d\sigma = I + \int_{\sigma=0}^{t} A + A^2 \sigma d\sigma \\
&= I + At + \frac{1}{2} A^2 t^2 \\
M_3(t,0) &= I + \int_{\sigma=0}^{t} A M_2(\sigma,0) d\sigma = I + \int_{\sigma=0}^{t} A(I + A\sigma + \frac{1}{2} A^2 \sigma) d\sigma \\
&= I + At + \frac{1}{2} A^2 t^2 + \frac{1}{2 \cdot 3} A^3 t^3
\end{aligned}
$$

Continuing in this manner we can write for $k \geq 1$:

$$
M_k(t,0) = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{2 \cdot 3} A^3 t^3 + \cdots + \frac{1}{k!} A^k t^k
$$

Step 2: $M_k(t,0)$ is a sum of $n \times n$ matrices and the (i,j) element of its $k+1$ term can be bounded as follows for any t in $0 \leq t \leq t_1$ (employing D/F/S 6):

$$
\begin{aligned}
|E_{i,j}(\frac{1}{k!} A^k t^k)| &\leq \frac{1}{k!} t^k |E_{i,j}(A^k)| \\
&\leq \frac{1}{k!} t^k n^{k-1} \alpha^k \\
&\leq \frac{1}{k!} t_1^k n^{k-1} \alpha^k
\end{aligned}
$$

So each element of $\frac{1}{k!} A^k t^k$ is bounded from above by the constant $\frac{1}{k!} t_1^k n^{k-1} \alpha^k$.

Step 3: This allows us to employ the Wierstrass M-test. Consider the series $\{K_k\}$:

$$
\begin{aligned}
& 1 + \alpha t_1 + \frac{n(\alpha t_1)^2}{2!} + \frac{n^2(\alpha t_1)^3}{3!} + \cdots \\
&= 1 + \frac{1}{n}(n\alpha t_1 + \frac{n^2(\alpha t_1)^2}{2!} + \frac{n^3(\alpha t_1)^3}{3!} + \cdots) \\
&= 1 + \frac{1}{n}(e^{n\alpha t_1} - 1)
\end{aligned}
$$

As we see the series of constants converges which implies that the sequence $\{M_k(t,0)$ converges uniformly on T. We call this limit the "transition matrix."

$$\Phi(t,0) = lim_{k\to\infty} M_k(t,0) = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

It is a special case of the Peano-Baker series (A here a constant matrix), and we also denote it as :

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

and define it as the "matrix exponential." Now, differentiating with respect to time term by term we obtain:

$$\begin{aligned}\frac{de^{At}}{dt} &= 0 + A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{2\cdot 3}A^4t^3 + \cdots \\ &= A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{2\cdot 3}A^3t^3 + \cdots) \\ &= Ae^{At}\end{aligned}$$

and where $e^{A\cdot 0} = \Phi(0,0) = I$.

Furthermore,

$$\frac{de^{At}x_0}{dt} = A\,\underbrace{e^{At}x_0}_{x(t)}$$

$$(\star) \qquad \dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

Therefore, $x(t) = e^{At}x_0$ is the solution of $(\star)$ above. We also can prove (not here) that this solution is unique.

Note: Since this is a "time-invariant" differential equation (A is constant), if the initial time is not 0 but rather t_0, the solution to $(\star)$ is given by:

$$x(t) = \Phi(t,t_0)x_0 = e^{A(t-t_0)}x_0$$

An Important Property of $\Phi(t,t_0)$, for arbitrary t_0, t_1, t.

$$\Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0)$$

Proof: The unique solution of:

$$(\star) \qquad \dot{x}(t) = Ax(t), \qquad x(t_0) = x_0$$

is given by $\bar{x}(t) = \Phi(t,t_0)x_0$. Suppose that at time $t_1, \bar{x}(t_1) = x_1$. Consider again equation $(\star)$ and develop its solution for initial condition x_1 at time t_1. In particular, we can write: $\bar{\bar{x}}(t) = \Phi(t,t_1)x_1$. Since the solution to

$(\star)$ is unique (both solutions pass from x_1 at time t_1) we must have for all t, t_1, t_0 and all x_0 that:

$$\bar{x}(t) = \bar{\bar{x}}(t)$$

This implies:

$$\begin{aligned} \Phi(t, t_0)x_0 &= \Phi(t, t_1)x_1 \\ \Phi(t, t_0)x_0 &= \Phi(t, t_1)\Phi(t_1, t_0)x_0 \\ \Phi(t, t_0) &= \Phi(t, t_1)\Phi(t_1, t_0) \end{aligned}$$

In particular, let $t = t_0$, and $t_0 = 0, t_1 = 1$. This becomes:

$$\begin{aligned} &\underbrace{\Phi(0,0)}_{I} = \underbrace{\Phi(0,1)}_{e^{-A}}\underbrace{\Phi(1,0)}_{a^{A}} \\ &e^{-A}e^{A} = I \end{aligned}$$

and we conclude that e^A **is invertible for any constant matrix A**. This is a very important result that we state as a Theorem:

Theorem 1: Le A be some $n \times n$ real matrix. Then $e^A \triangleq \exp(A)$ is an invertible matrix:

$$\exp(A) : A \quad \longrightarrow e^A \in \mathrm{G}\ell(n, \mathbb{R})$$

Let the set of $n \times n$ skew symmetric matrices (i.e., $A = -A^T$) be denoted by $so(n)$.

Proposition: Let $A \in so(n)$. Then e^A is an orthogonal matrix (i.e., $e^A \cdot (e^A)^T = I$.

proof: From Theorem 1 above we know that e^A is invertible. In fact,

$$e^A \underbrace{e^{-A}}_{(e^A)^{-1}} = I$$

since $-A = A^T$

$$e^A \cdot e^{A^T} = e^A \cdot (e^A)^T = I$$

directly from the Peano-Baker series.