ME/SE 740

Lecture 5

3-D Rigid Body Motions and Coordinate Transformations

Summary from last lecture:

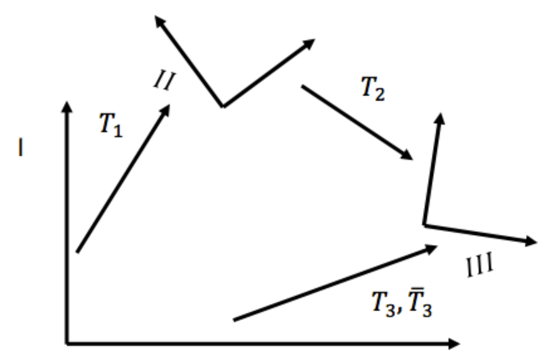

Figure 1: **Rigid Motions**

Dual interpretation of T:

$$T: \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix}$$

- Coordinate Transformation
- Rigid Body Motion

Algebraic law of composition:

$$S: \quad \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \qquad T: \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T \circ S: \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

For coordinate transformations: $T_3 = T_1 \circ T_2$.
For rigid body motions: $\bar{T}_3 = T_2 \circ T_1$.

Consider now the the planar kinematic chain shown below:

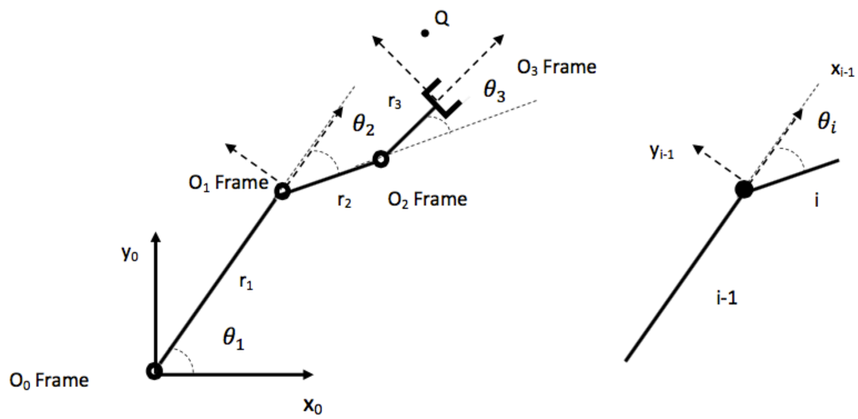

Figure 2: **3-Link Kinematic Chain**

The coordinate frames O_0, O_1, O_2, O_3 are all right-handed (positive direction of angles is couter-clockwise).

The orientation of the end effector frame is:

$$\begin{pmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{pmatrix}$$

The position of the origin of frame 3 relative to the predecessor frame is:

$$\begin{pmatrix} r_3\cos\theta_3 \\ r_3\sin\theta_3 \end{pmatrix}$$

Working backwards the position and orientation of point "Q" and the tool frame with respect to joint 2 which is at the distal end of link 1, is:

$$\begin{pmatrix} \cos(\theta_2+\theta_3) & -\sin(\theta_2+\theta_3) \\ \sin(\theta_2+\theta_3) & \cos(\theta_2+\theta_3) \end{pmatrix}, \quad \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}\begin{pmatrix} r_3\cos\theta_3 \\ r_3\sin\theta_3 \end{pmatrix} + \begin{pmatrix} r_2\cos\theta_2 \\ r_2\sin\theta_2 \end{pmatrix}$$

Working all the way back to the base frame (0-frame) the position and orientation of Q and the tool frame is:

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}\begin{pmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}\left[\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}\left\{\begin{pmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{pmatrix}\begin{pmatrix} r_3 \\ 0 \end{pmatrix} + \begin{pmatrix} r_2 \\ 0 \end{pmatrix}\right\} + \begin{pmatrix} r_1 \\ 0 \end{pmatrix}\right]$$

Which can be expressed as:

$$\left(\begin{array}{cc} \cos(\theta_1+\theta_2+\theta_3) & -\sin\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) \end{array} \right), \quad \left(\begin{array}{c} r_1\cos\theta_1 \\ r_1\sin\theta_1 \end{array} \right) + \left(\begin{array}{c} r_2\cos(\theta_1+\theta_2) \\ r_2\sin(\theta_2+\theta_2) \end{array} \right) + \left(\begin{array}{c} r_3\cos(\theta_1+\theta_2+\theta_3) \\ r_3\sin(\theta_1+\theta_2+\theta_3) \end{array} \right)$$

The next question we would like to explore is what happens in three dimensions. Rotation in the plane can be thought of as a mapping :

$$\left(\begin{array}{c} x \\ y \end{array} \right) \longmapsto \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right)$$

We can think of this as a mapping in 3-space expressed as:

$$\left(\begin{array}{c} x \\ y \\ z \end{array} \right) \longmapsto \underbrace{\left(\begin{array}{ccc} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array} \right)}_{R(z,\theta)} \left(\begin{array}{c} x \\ y \\ z \end{array} \right)$$

Where $R(z,\theta)$ is rotation about the z-axis, also referred to as "yaw."

Rotation about the y-axis $R(y,\theta)$, also referred to as "pitch" is expressed as:

$$\left(\begin{array}{c} x \\ y \\ z \end{array} \right) \longmapsto \underbrace{\left(\begin{array}{ccc} \cos\theta & 0 & \sin\theta \\ 0 & 1 & \\ -\sin\theta & 0 & \cos\theta \end{array} \right)}_{R(y,\theta)} \left(\begin{array}{c} x \\ y \\ z \end{array} \right)$$

Rotation about the x-axis $R(x,\theta)$, also referred to as "roll" is expressed as:

$$\left(\begin{array}{c} x \\ y \\ z \end{array} \right) \longmapsto \underbrace{\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{array} \right)}_{R(x,\theta)} \left(\begin{array}{c} x \\ y \\ z \end{array} \right)$$

Rotations in 2-D commute. Rotations in 3-D do NOT commute (in general).

Consider the net rotation of a 90° rotation about the z-axis, followed by a 90° rotation about the y-axis:

$$\left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) \left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right)$$

Compare this with rotation of 90° rotation about the y-axis, followed by a 90° rotation about the z-axis:

$$\left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array} \right)$$

Clearly, the right-hand-sides of the above two equations are NOT equal.

If A, B are 3×3 proper (right hand coordinate system) rotation matrices it is generally the case that $AB \neq BA$.

In general, coordinate transformations or rigid body motions in space are represented by pairs:

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

Consider the rotation in 3-D shown in the following figure:

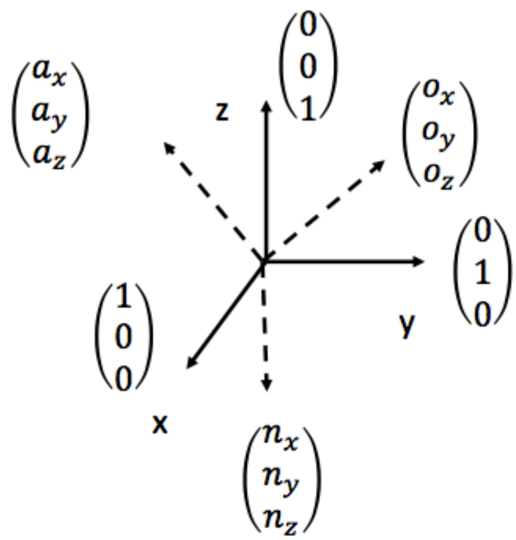

Figure 3: **3-D Rotation**

There are 9 parameters hat characterize this rotation and there are 6 constraints that these parameters satisfy. Therefore this leaves $9 - 6$ "degrees of freedom:"

$$\begin{aligned} n_x^2 + n_y^2 + n_z^2 &= 1 \\ o_x^2 + o_y^2 + o_z^2 &= 1 \\ a_x^2 + a_y^2 + a_z^2 &= 1 \end{aligned}$$

$$\begin{aligned} n_x o_x + n_y o_y + n_z o_z &= 0 \\ n_x a_x + n_y a_y + n_z a_z &= 0 \\ o_x a_x + o_y a_y + o_z a_z &= 0 \end{aligned}$$

An important consequence of the orthonormality relations above is the following:

$$\underbrace{\begin{pmatrix} n_x & n_y & n_z \\ o_x & o_y & o_z \\ a_x & a_y & a_z \end{pmatrix}}_{R^T} \underbrace{\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}}_{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

That is, if R is a proper 3×3 rotation matrix then $R^T R = I$. This implies that R^T is the left inverse of R, but a very simple Linear Algebra calculation shows that any left inverse is a right inverse and hence $RR^T = I$, and that $R^T = R^{-1}$.

Inverse transformations. Consider the composition of two rigid transformations:

$$\begin{pmatrix} n_x^2 & o_x^2 & a_x^2 \\ n_y^2 & o_y^2 & a_y^2 \\ n_z^2 & o_z^2 & a_z^2 \end{pmatrix} \begin{pmatrix} n_x^1 & o_x^1 & a_x^1 \\ n_y^1 & o_y^1 & a_y^1 \\ n_z^1 & o_z^1 & a_z^1 \end{pmatrix}, \quad \begin{pmatrix} n_x^2 & o_x^2 & a_x^2 \\ n_y^2 & o_y^2 & a_y^2 \\ n_z^2 & o_z^2 & a_z^2 \end{pmatrix} \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} + \begin{pmatrix} p_x^2 \\ p_y^2 \\ p_z^2 \end{pmatrix}$$

Suppose this is equal to the identity transformation (no rotation, no translation):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This means that:

$$\begin{pmatrix} n_x^2 & o_x^2 & a_x^2 \\ n_y^2 & o_y^2 & a_y^2 \\ n_z^2 & o_z^2 & a_z^2 \end{pmatrix} = \begin{pmatrix} n_x^1 & o_x^1 & a_x^1 \\ n_y^1 & o_y^1 & a_y^1 \\ n_z^1 & o_z^1 & a_z^1 \end{pmatrix}^{-1} = \begin{pmatrix} n_x^1 & o_x^1 & a_x^1 \\ n_y^1 & o_y^1 & a_y^1 \\ n_z^1 & o_z^1 & a_z^1 \end{pmatrix}^{T}$$

and

$$\begin{pmatrix} p_x^2 \\ p_y^2 \\ p_z^2 \end{pmatrix} = - \begin{pmatrix} n_x^2 & o_x^2 & a_x^2 \\ n_y^2 & o_y^2 & a_y^2 \\ n_z^2 & o_z^2 & a_z^2 \end{pmatrix} \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} = - \begin{pmatrix} n_x^1 & n_y^1 & n_z^1 \\ o_x^1 & o_y^1 & o_z^1 \\ a_x^1 & a_y^1 & a_z^1 \end{pmatrix} \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix}$$

which is analogus to the 2-D case. In particular the "inverse" of

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}, \quad \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}, \qquad is \qquad \begin{pmatrix} n_x & n_y & n_z \\ o_x & o_y & o_z \\ a_x & a_y & a_z \end{pmatrix}, \quad - \begin{pmatrix} n_x & n_y & n_z \\ o_x & o_y & o_z \\ a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

It will be convenient to adopt short-hand matrix notation:

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \longleftrightarrow R, \qquad \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \longleftrightarrow \vec{r}$$

Spatial rotations and their composition can be tought of as:

$$(R, \vec{r}), (S, \vec{s})$$
$$(SR, S\vec{r} + \vec{s}) = (S, \vec{s}) \circ (R, \vec{r})$$

A wonderfully useful fact is that we can associate rigid transformations or motions with 4×4 matrices:

$$(R, \vec{r}) \quad \longleftrightarrow \quad \begin{pmatrix} R & \vec{r} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which implies that composition of rigid body motions can be represented by matrix multiplication:

$$(S, \vec{s}) \circ (R, \vec{r}) \longleftrightarrow \begin{pmatrix} S & \vec{s} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & \vec{r} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} SR & S\vec{r} + \vec{s} \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} n_x & n_y & n_z & -\vec{n} \cdot \vec{p} \\ o_x & o_y & o_z & -\vec{o} \cdot \vec{p} \\ a_x & a_y & a_z & -\vec{a} \cdot \vec{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$