ME/SE 740

Lecture 23

Introduction to Lagrangian Mechanics

An idealized model of a robot has n-masses (point masses) interconnected by a set of links:

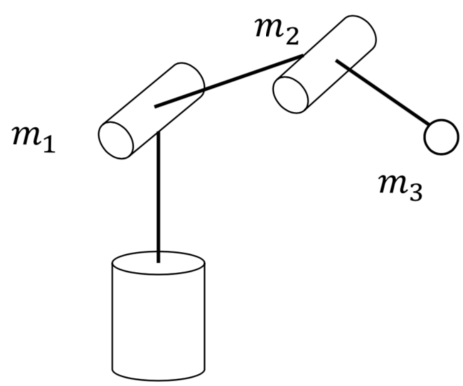

Figure 1: **Idealized Three Link Manipulator**

Let $(x_i \; y_i \; z_i)^T$ be the coordinates of the i-th point mass. Each

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \begin{pmatrix} x_i(\theta_1, \theta_2, \dots, \theta_n) \\ y_i(\theta_1, \theta_2, \dots, \theta_n) \\ z_i(\theta_1, \theta_2, \dots, \theta_n) \end{pmatrix}$$

is a function of the joint angles θ_i. Let X_i, Y_i, Z_i, be the components of the total force acting on the i-th point mass. Then we have:

$$m_i \begin{pmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{z}_i \end{pmatrix} = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$$

Multiplying both sides of theses equations by:

$$\begin{pmatrix} \frac{\partial x_i}{\partial \theta_j} \\ \frac{\partial y_i}{\partial \theta_j} \\ \frac{\partial z_i}{\partial \theta_j} \end{pmatrix}$$

we obtain (on summing):

$$(\star) \qquad \sum_i m_i(\ddot{x}_i \frac{\partial x_i}{\partial \theta_j} + \ddot{y}_i \frac{\partial y_i}{\partial \theta_j} + \ddot{z}_i \frac{\partial z_i}{\partial \theta_j}) = \sum_i (X_i \frac{\partial x_i}{\partial \theta_j} + Y_i \frac{\partial y_i}{\partial \theta_j} + Z_i \frac{\partial z_i}{\partial \theta_j})$$

But

$$\frac{\partial \dot{x}_i}{\partial \dot{\theta}_j} = \frac{\partial}{\partial \dot{\theta}_j}(\frac{\partial x_i}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial x_i}{\partial \theta_2}\dot{\theta}_2 + \ldots + \frac{\partial x_i}{\partial \theta_n}\dot{\theta}_n) = \frac{\partial x_i}{\partial \theta_j}$$

Hence

$$\begin{aligned}
\ddot{x}_i \frac{\partial x_i}{\partial \theta_j} &= \ddot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{\theta}_j} \\
&= \frac{d}{dt}(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{\theta}_j}) - \dot{x}_i \frac{d}{dt}(\frac{\partial x_i}{\partial \theta_j}) \\
&= \frac{d}{dt}(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{\theta}_j}) - \dot{x}_i(\frac{\partial^2 x_i}{\partial \theta_j \partial \theta_1}\dot{\theta}_1 + \ldots + \frac{\partial^2 x_i}{\partial \theta_j \partial \theta_n}\dot{\theta}_n) \\
&= \frac{d}{dt}(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{\theta}_j}) - \dot{x}_i \frac{\partial}{\partial \theta_j}(\dot{x}_i) \\
&= \frac{d}{dt}(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{\theta}_j}) - \dot{x}_i \frac{\partial \dot{x}_i}{\partial \theta_j} \\
&= \frac{d}{dt}\{\frac{\partial}{\partial \dot{\theta}_j}(\frac{1}{2}\dot{x}_i^2)\} - \frac{\partial}{\partial \theta_j}(\frac{1}{2}\dot{x}_i^2)
\end{aligned}$$

Returning to $(\star)$ we find that:

$$\begin{aligned}
\sum_i m_i(\ddot{x}_i \frac{\partial x_i}{\partial \theta_j} + \ddot{y}_i \frac{\partial y_i}{\partial \theta_j} + \ddot{z}_i \frac{\partial z_i}{\partial \theta_j}) &= \sum_i \{\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}_j}(\frac{1}{2}m_i\dot{x}_i^2) - \frac{\partial}{\partial \theta_j}(\frac{1}{2}m_i\dot{x}_i^2) \\
&\quad + \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}_j}(\frac{1}{2}m_i\dot{y}_i^2) - \frac{\partial}{\partial \theta_j}(\frac{1}{2}m_i\dot{y}_i^2) \\
&\quad + \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}_j}(\frac{1}{2}m_i\dot{z}_i^2) - \frac{\partial}{\partial \theta_j}(\frac{1}{2}m_i\dot{z}_i^2)\} \\
(\star\star) \qquad &= \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}_j}(\sum_i \frac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)) - \frac{\partial}{\partial \dot{\theta}_j}(\sum_i \frac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2))
\end{aligned}$$

We next observe that all terms in the expression are functions of $\theta_1, \theta_2, \ldots, \theta_n, \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n$. Also $x_i = x_i(\theta_1, \ldots, \theta_n)$. Hence:

$$\begin{aligned}
\dot{x}_i &= \frac{d}{dt}(x_i(\theta_1, \ldots, \theta_n)) \\
&= \frac{\partial x_i}{\partial \theta_1}\dot{\theta}_1 + \ldots + \frac{\partial x_i}{\partial \theta_n}\dot{\theta}_n
\end{aligned}$$

Similarly,

$$\dot{y}_i = \frac{\partial y_i}{\partial \theta_1}\dot{\theta}_1 + \ldots + \frac{\partial y_i}{\partial \theta_n}\dot{\theta}_n$$
$$\dot{z}_i = \frac{\partial z_i}{\partial \theta_1}\dot{\theta}_1 + \ldots + \frac{\partial z_i}{\partial \theta_n}\dot{\theta}_n$$

Hence,

$$\sum_i^n \frac{1}{2} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = \text{kinetic energy of the system} = T(\theta_1, \theta_2, \ldots, \theta_n, \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)$$

The expression $(\star\star)$ is just

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j}$$

and equation $(\star)$ becomes:

$$\star' \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j} = \sum_{i=1}^n \left(\frac{\partial x_i}{\partial \theta_j} \quad \frac{\partial y_i}{\partial \theta_j} \quad \frac{\partial z_i}{\partial \theta_j}\right) \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$$

This expression will be distilled still further but to do this we need an amazing fact about transmission of forces through a linkage. Consider the position relationship:

$$\vec{x}_i = f_i(\theta_1, \ldots, \theta_n)$$

(for the i-th particle in our system). Suppose there are no forces acting on the system except a force $\vec{F}_i$ on $\vec{x}_i$. Let this force move the system by an amount $\delta\vec{x}_i$ corresponding to (infinitesimal) movement $\delta\theta_j$, for $(j = 1, \ldots, n)$ of each joint. Then with $\vec{\tau}_j$ being the torques induced in the joints we have:

$$\underbrace{\delta\vec{x}_i \cdot \vec{F}_i}_{\text{work measured w.r.t the i-th particle}} \quad = \quad \underbrace{\sum_j \delta\theta_j \vec{\tau}_j}_{\substack{\text{total work from joint motions} \\ \text{(where } \vec{\tau}_j \text{ are torques induced in the joints)}}}$$

This equation may be re-written as:

$$\delta\vec{x}_i \cdot \vec{F}_i = \vec{\tau} \cdot \delta\vec{\theta}$$

Now

$$\delta\vec{x}_i = \frac{\partial f_i}{\partial \theta} \cdot \delta\vec{\theta}$$

$$\Longrightarrow$$

$$\delta\vec{x}_i \cdot \vec{F}_i = \delta\vec{\theta}^{\,T} (\frac{\partial f_i}{\partial \theta})^T \cdot \vec{F}_i$$

Hence, we have:

$$\delta\vec{\theta}^{\,T} (\frac{\partial f_i}{\partial \theta})^T \cdot \vec{F}_i = \delta\vec{\theta}^{\,T} \cdot \vec{\tau}$$

This equation is valid for <u>any</u> "imaginary" infinitesimal displacement $\delta\vec{\theta}$ (PRINCIPLE OF VIRTUAL WORK).

Hence:

$$(\frac{\partial f_i}{\partial \theta})^T \cdot \vec{F}_i = \vec{\tau}$$

Consider now the manipulator shown below where $x = f(\theta)$ and the Jacobian is given by $J = \frac{\partial f}{\partial \theta}$. What the manipulator Jacobian tells us is the following:

$$\dot{x} = J\dot{\theta}$$

how velocities are transmitted from joints to the end-effector, and

$$\vec{\tau} = J^T \vec{F}$$

how a force vector at the end-effector is felt as torques at the joints.

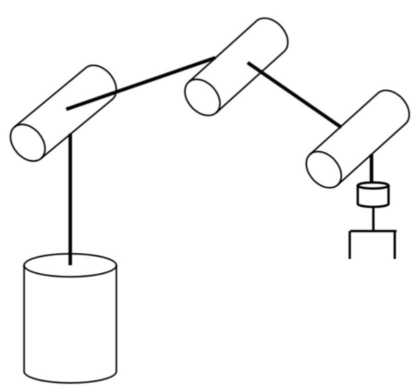

Figure 2: **Manipulator with End-Effector**

What about singularities in J^T? For effective transmission of forces, singularities are a <u>good</u> thing.

From above we have:

$$
\begin{aligned}
m\begin{pmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{z}_i \end{pmatrix} &= \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} \\
\sum_i m_i(\ddot{x}_i \frac{\partial x_i}{\partial \theta_j} + \ddot{y}_i \frac{\partial y_i}{\partial \theta_j} + \ddot{z}_i \frac{\partial z_i}{\partial \theta_j}) &= \sum_i (X_i \frac{\partial x_i}{\partial \theta_j} + Y_i \frac{\partial y_i}{\partial \theta_j} + Z_i \frac{\partial z_i}{\partial \theta_j}) \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j} &= \sum_i (X_i \frac{\partial x_i}{\partial \theta_j} + Y_i \frac{\partial y_i}{\partial \theta_j} + Z_i \frac{\partial z_i}{\partial \theta_j})
\end{aligned}
$$

and

$$\dot{x}_i = \frac{\partial x_i}{\partial \theta_1}\dot{\theta}_1 + \ldots + \frac{\partial x_i}{\partial \theta_n}\dot{\theta}_n$$

where the kinetic energy is given by:

$$T = \sum_i^n \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

This can be expressed as:

$$T(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{2}(\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n) M(\theta_1, \theta_2, \ldots, \theta_n) \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{pmatrix}$$

where $M(\theta_1, \theta_2, \ldots \theta_n)$ is a symmetric, positive semidefinite (the components of which are left to you to work out as an exercise).

In terms of the notation for the problem at hand we have:

$$
\begin{aligned}
f_i(\theta) &= \begin{pmatrix} x_i(\theta) \\ y_i(\theta) \\ z_i(\theta) \end{pmatrix} = \begin{pmatrix} x_i(\theta_1, \theta_2, \ldots, \theta_n) \\ y_i(\theta_1, \theta_2, \ldots, \theta_n) \\ z_i(\theta_1, \theta_2, \ldots, \theta_n) \end{pmatrix} \\
\frac{\partial f_i}{\partial \theta} &= \begin{pmatrix} \frac{\partial x_i}{\partial \theta_1} & \frac{\partial x_i}{\partial \theta_2} & \ldots & \frac{\partial x_i}{\partial \theta_n} \\ \frac{\partial y_i}{\partial \theta_1} & \frac{\partial y_i}{\partial \theta_2} & \ldots & \frac{\partial y_i}{\partial \theta_n} \\ \frac{\partial z_i}{\partial \theta_1} & \frac{\partial z_i}{\partial \theta_2} & \ldots & \frac{\partial z_i}{\partial \theta_n} \end{pmatrix}
\end{aligned}
$$

The force relation is:

$$\vec{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix} = \begin{pmatrix} \frac{\partial x_i}{\partial \theta_1} & \frac{\partial y_i}{\partial \theta_1} & \frac{\partial z_i}{\partial \theta_1} \\ \frac{\partial x_i}{\partial \theta_2} & \frac{\partial y_i}{\partial \theta_2} & \frac{\partial z_i}{\partial \theta_2} \\ \vdots & \vdots & \vdots \\ \frac{\partial x_i}{\partial \theta_n} & \frac{\partial y_i}{\partial \theta_n} & \frac{\partial z_i}{\partial \theta_n} \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$$

(Actually, there should be a superscript or something on the τ_i's to record the fact that these are torques corresponding to the force acting on the i-th point mass.)

Returning to ($\star'$) we have:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_j} - \frac{\partial T}{\partial \theta_j} = \sum_{i=1}^{n} (\frac{\partial x_i}{\partial \theta_j} \quad \frac{\partial y_i}{\partial \theta_j} \quad \frac{\partial z_i}{\partial \theta_j}) \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} = \sum_{i=1}^{n} \tau_j^i = Q_j$$

where τ_j^i is the force/torque at the j-th joint corresponding to the net force applied to the i-th point mass, and Q_j is the net force/torque on the j-th joint.

Theorem 1: Suppose the configuration of a dynamical system may be specified by coordinates $\theta_1, \theta_2, \ldots, \theta_n$, and suppose the kinetic energy (corresponding to any possible motion) may be written $T = T(\theta_1, \theta_2, \ldots, \theta_n; \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)$, then the equations of motion for this system are given by:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_i} - \frac{\partial T}{\partial \theta_i} = Q_i \qquad (i = 1, \ldots, n)$$

where Q_i is the i-th generalized force acting on the system.

Remark 1: In robotics, this formulation of Newton's 2$^{\text{nd}}$ Law is quite natural, since torques and forces from actuators are applied to the joints.

Remark 2: There may be a potential energy function $V(\theta_1, \theta_2, \ldots, \theta_n)$ denoting the path independent work done in moving the system from some reference configuration $(\theta_1^\star, \theta_2^\star, \ldots, \theta_n^\star)$ to $(\theta_1, \theta_2, \ldots, \theta_n)$. This potential energy gives rise to conservative generalized forces $Q_j = -\frac{\partial V}{\partial \theta_j}$.

Theorem 2: Suppose the configuration of a dynamical system may be specified by coordinates $\theta_1, \theta_2, \ldots, \theta_n$, and suppose kinetic energy (corresponding to any possible motion) may be written $T = T(\theta_1, \theta_2, \ldots, \theta_n; \dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)$, and the potential energy (corresponding to any possible configuration) may be written $V = V(\theta_1, \theta_2, \ldots, \theta_n)$, then the equations of motion for this system are given by:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_i} - \frac{\partial T}{\partial \theta_i} = -\frac{\partial V}{\partial \theta_i} + \tau_i \qquad (i = 1, \ldots, n)$$

where τ_i is the i-th generalized (exogenous) force affecting the variable θ_i.

Corollary: If we define the Lagrangian $L = T - V$, then Lagrange's equations of motion (or the Euler-Lagrange equations) may be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i \qquad (i = 1, \ldots, n)$$

Hamilton's Interpretation: Principle of Least Actions

Define the action:

$$\begin{aligned} S &= \int_{t_0}^{t_1} L dt \\ &= \int_{t_0}^{t_1} T - V dt \end{aligned}$$

Then $\delta S = 0$, where $\delta S =$ variation in S that occurs when we vary the coordinates of the path $(\theta, \dot{\theta})$ by an infinitesimal amount. When variation about given path is thus $= 0$, the value of S along the path may be a local maximum or minimum in the set of all paths having the same beginning and end points. <u>This is like a first derivative test</u>.

$$\begin{aligned} S + \delta S &= \int_{t_o}^{t_1} L(\theta + \delta\theta, \dot{\theta} + \delta\dot{\theta}) dt \\ &= \int_{t_o}^{t_1} L(\theta, \dot{\theta}) + \frac{\partial L}{\partial \theta}\delta\theta + \frac{\partial L}{\partial \dot{\theta}}\delta\dot{\theta} + \underbrace{o(\delta\theta)}_{\text{h. o. t. in } \delta\theta,\ \delta\dot{\theta}} \quad dt \\ &= \int_{t_o}^{t_1} L(\theta, \dot{\theta}) + \frac{\partial L}{\partial \theta}\delta\theta - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}\delta\theta + o(\delta\theta) dt \end{aligned}$$

This last equation comes from the fact that:

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{\theta}}\delta\dot{\theta} dt = \frac{\partial L}{\partial \dot{\theta}}\delta\theta \Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}\delta\theta dt$$

where that first term on the right is zero because $\theta(t)$ and $\theta(t) + \delta\theta(t)$ have the same endpoints at t_0, t_1.

Now suppose $(\theta(t), \dot{\theta(t)})$ defines a curve minimizing S among all possible curves taking on prescribed end point values at t_0, t_1. Then along this curve:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

Otherwise, let

$$\delta\theta(t) = \epsilon\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}\right) \qquad (\epsilon > 0)$$

then

$$S + \delta S = \int_{t_0}^{t_1} L(\theta, \dot{\theta}) dt - \int_{t_0}^{t_1} \epsilon\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}\right)^2 dt + o(\epsilon)$$

For $\epsilon > 0$ sufficiently small, $S + \delta S < S$, contradicting the assumed local minimality of S.

Therefore, trajectories which minimize the action $\int_{t_0}^{t_1} L(\theta, \dot{\theta}) dt$ satisfy the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$