ME/SE 740

Lecture 22

Extended Jacobian

EXTENDED JACOBIAN APPROACH TO RESOLUTION OF REDUNDANCY:

If redundancy is resolved by means of a functional constraint $G(\theta) = 0$, we define the extended Jacobian:

$$J_e = \begin{pmatrix} J \\ \\ \frac{\partial G}{\partial \theta} \end{pmatrix}$$

Before we proceed, let us return to the last example in Lecture 21 (i.e., the three link manipulator) and develop J_e when $G(\theta) = sin^2\theta_2 + sin^2\theta_3$. The Extended Jacobian is given below, followed by two plots: the first is a 3-D plot of $\det J_e = 2(s_{23}(s_3c_3 - s_2c_2) + s_2s_3(c_3 - c_2))$; and the second shows link angles θ_2, θ_3 that make $\det J_e = 0$.

$$J_e \quad = \quad \begin{pmatrix} -s_1 - s_{12} - s_{123} & -s_{12} - s_{123} & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \\ 0 & 2s_2c_2 & 2s_3c_3 \end{pmatrix}$$

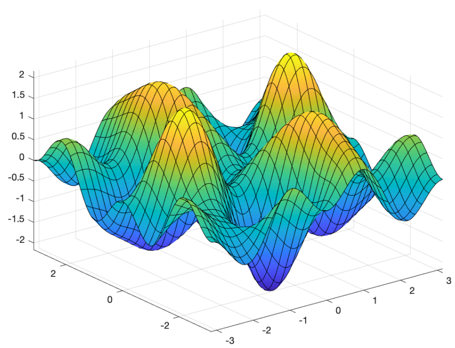


Figure 1: **3-D Plot of** $detJ_e$

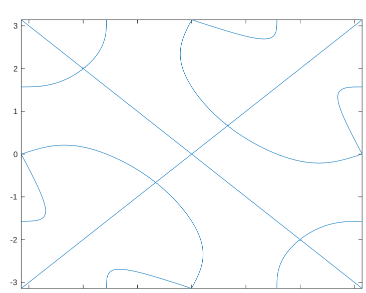


Figure 2: θ_2, θ_3 **that make det** J_e **equal to zero**

Note that along trajectories satisfying the constraint that:

$$\begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} = J_e \dot{\theta}$$

the joint space trajectories are formally given by the differential equation:

$$\dot{\theta} = J_e^{-1} \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix}$$

But to make this rigorous we need to know when J_e will be singular. Certainly whenever $J = \frac{\partial f}{\partial \theta}$ is singular, but by proper choice of G, such mechanical singularities can be avoided. Are there other singularities?

$$J_e = \begin{pmatrix} J \\ G_\theta \end{pmatrix}$$

where the rows comprising G_θ are partial derivatives of the constraint function. For simplicity we will restrict our discussion to the case $m = n - 1$, degree 1 redundancy.

Now

$$\begin{aligned} J_e(J^\dagger \vdots \vec{n}) &= \begin{pmatrix} J \\ G_\theta \end{pmatrix} (J^T (JJ^T)^{-1} \vdots \vec{n}) \\ &= \left(\begin{array}{c|c} I & 0 \\ \hline \xi_1 \ldots \xi_{n-1} & G_\theta \vec{n} \end{array} \right) \qquad (\star) \end{aligned}$$

where $J^\dagger = J^T(JJ^T)^{-1}$ and is an $n \times (n-1)$ matrix. Thus there are potential problems in inverting $J_e \Leftrightarrow G_\theta \cdot \vec{n} = 0 \Leftrightarrow$ (one of G_θ or $\vec{n}$ are zero or G_θ lies in the row space of J).

Note that $(\star)$ provides a formula whereby J_e^{-1} may be explicitly written as:

$$\begin{aligned} J_e^{-1} &= (J^\dagger \vdots \vec{n}) \left(\begin{array}{c|c} I & 0 \\ \hline -\frac{\xi_1}{G_\theta \vec{n}} \ldots -\frac{\xi_{n-1}}{G_\theta \vec{n}} & \frac{1}{G_\theta \vec{n}} \end{array} \right) \\ &= \left(J^\dagger - \frac{\vec{n} \cdot \xi^T}{G_\theta \vec{n}} \vdots \frac{\vec{n}}{G_\theta \vec{n}} \right) \end{aligned}$$

Now

$$\dot{\theta} = J_e^{-1} \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} = J^\dagger \dot{x} \; - \; \frac{\vec{n} \cdot \xi^T}{G_\theta \vec{n}} \dot{x}$$

and since

$$\xi^T = (\xi_1, \ldots, \xi_{n-1}) = G_\theta J^\dagger$$

we have

$$\dot{\theta} = J^\dagger \dot{x} - \frac{\vec{n} \cdot G_\theta J^\dagger \dot{x}}{G_\theta \cdot \vec{n}} = J^\dagger \dot{x} + v(t)\vec{n}$$

where

$$v(t) = -\frac{G_\theta J^\dagger \dot{x}}{G_\theta \cdot \vec{n}}$$

Consider the conditions where invertibility of J_e is in question:

$$G_\theta \cdot \vec{n} = 0$$

<u>Case:</u> $\vec{n} = 0$

$$\vec{n} = \lambda(J_1, \ -J_2, \ldots, (-1)^{n+1} J_n)$$

where λ is a scalar and J_k is the k-th principal minor of J

$$\vec{n} = 0 \quad \Leftrightarrow \text{rank} J < m \qquad \Leftrightarrow \text{configuration is kinematically singular}$$

<u>Case:</u> $G_\theta = 0$

In this case, numerator and denominator entries can cancel in

$$v(t) = -\frac{G_\theta J^\dagger \dot{x}}{G_\theta \cdot \vec{n}}$$

resulting in the potential singularity being ignorable.

<u>Case:</u> $\vec{n} \neq 0,\ G_\theta \neq 0$

This is a truly singular situation. We have G_θ belonging to the row space of J. The satisfaction of the constraint requires $G_\theta \dot{\theta} \equiv 0$. But $G_\theta = y^T J$ for some m-vector y. Hence we must have

$$y^T J \dot{\theta} = 0$$

and thus $y^T \dot{x} = 0$ since $\dot{x} = J\dot{\theta}$. Therefore, we cannot move the end-effector in the direction y <u>and</u> continue to satisfy the constraint.

<u>Definition:</u> Configurations corresponding to $G_\theta \neq 0, \ \ \vec{n} \neq 0$ but $G_\theta \cdot \vec{n} = 0$ are called <u>algorithmic singularities</u>.

<u>Example</u>

Consider the planar manipulator

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix}$$

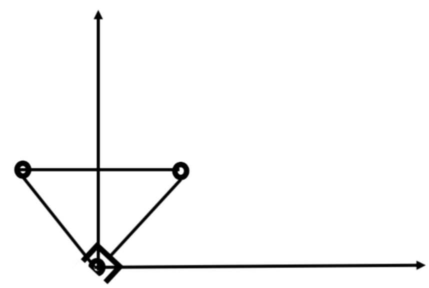
The constraint maximizing $sin^2\theta_2 + sin^2\theta_3$ is $\theta_2 = \theta_3$. With this constraint, if we examine

$$\frac{\frac{\partial G}{\partial \theta} \cdot J^\dagger}{\frac{\partial G}{\partial \theta} \cdot \vec{n}}$$

we find that in lowest terms, the common denominator of the entries in this vector is:

$$(sin\theta_2 + sin(2\theta_2))|\vec{n}|^2$$

This vanishes when $|\vec{n}|^2 = 0$ (not of interest) and when $\theta = \pm\frac{2\pi}{3}$. This is a true algorithmic singularity wherein the manipulator places its end-effector on its base.



For redundant manipulators:

1. Brockett showed that in general we cannot rule out the possibility of singularities.

2. While instantaneously optimizing a figure of merit regarding configurations provides a useful approach to resolution of kinematic redundancy, in general any such approach will be associated with algorithmic singularity.

Other work on the resolution of kinematic redundancy:

Pathwise Resolution of Kinematic Redundancy

Consider the following expression:

$$\int_0^T (\frac{1}{2}\dot{\theta}^T W^{-1}\dot{\theta} + g(\theta))dt \qquad (A)$$

<u>**Theorem**</u> Joint space trajectories which optimize (A) satisfy

$$\ddot{\theta} = J_W^{\dagger}(\ddot{x} - \dot{J}\dot{\theta}) + P_W(\dot{W}W^{-1}\dot{\theta} + Wg_{\theta}) \qquad (\star\star)$$

where

$$P_W = (I - J_W^{\dagger}J)$$

is the weighted null space projection operator of the Jacobian, J, and

$$J_W^\dagger = WJ^T(JWJ^T)^{-1}$$

and $J_W^\dagger$ is the weighted pseudo-inverse of J.

Proof: This theorem follows from a variational argument in which the joint space trajectory variations $\delta\theta$ are constrained to lie in the direction of the null space of J. The Euler-Lagrange operator defined by the functional (A) is:

$$\frac{d}{dt}(W^{-1}\dot{\theta}) - \frac{\partial g}{\partial \theta}$$

and if $\theta(\cdot)$ minimizes (A) over all trajectories which vary with respect to $\theta(\cdot)$ in the direction $ker J(\theta(t))$ (i.e., with respect to all trajectories corresponding to the given operational space path $x(\cdot)$), it follows that $\theta(\cdot)$ must satisfy

$$P_W\{W[\frac{d}{dt}(W^{-1}\dot{\theta}) - \frac{\partial g}{\partial \theta}]\} = 0 \qquad (B)$$

Note that P_W is the orthogonal projection onto the null space of J to the inner product defined by the symmetric positive definite matrix W^{-1}. If we differentiate $\dot{x} = J\dot{\theta}$ with respect to t, we obtain

$$\ddot{x} = \dot{J}\dot{\theta} + J\ddot{\theta}$$

which may be equivalently written as

$$\ddot{\theta} = J_W^\dagger(\ddot{x} - \dot{J}\dot{\theta}) + P_W v \qquad (C)$$

for an appropriate choice of v. Multiplying both sides of this equation by the projection operator P_W and noting that $P_W J_W^\dagger = 0$, we obtain

$$P_W\ddot{\theta} = P_W v.$$

Using (B), we find that this implies

$$P_W v = \dot{W}W^{-1}\dot{\theta} + W\frac{\partial g}{\partial \theta}$$

and substituting this into (C) proves the theorem.

Typical Boundary Conditions:

I. Initial Value Conditions:

$$\begin{aligned} x(t_0) &= f(\theta(t_0)) \\ \dot{x}(t_0) &= J(\theta(t_0))\dot{\theta}(t_0) \end{aligned}$$

II. Two-point Boundary Values:

$$\begin{aligned} x(t_0) &= f(\theta(t_0)) \\ x(t_f) &= f(\theta(t_f)) \end{aligned}$$

III. Natural Boundary Conditions:

$$P_W \dot{\theta}(t_0)) = 0$$
$$P_W \dot{\theta}(t_f)) = 0$$

IV. Periodic Boundary Conditions: if x satisfies $x(0) = x(T)$, the objective is to find trajectories which satisfy $(\star\star)$ subject to $\theta(0) = \theta(T)$ and $\dot{\theta}(0) = \dot{\theta}(T)$

Homotopy Continuation Methods Applied to Path-wise Resolution of Kinematic Redundancy

Problem:

$$min \int_0^T \left(\frac{\epsilon}{2}\|\dot{\theta}\|^2 + (1-\epsilon)g(\theta)\right)dt$$

with

$$\begin{aligned} \theta(0) = \theta(T), \qquad & \dot{\theta}(0) = \dot{\theta}(T) \\ \epsilon\hat{n}\cdot\ddot{\theta} \quad = \quad & (1-\epsilon)G(\theta) \end{aligned}$$

where

$$G(\theta) = \frac{\partial g}{\partial \theta}(\theta)\cdot\hat{n}(\theta)$$

Reference: D. P. Martin, J. B., J. M. Hollerbach, "Resolution of kinematic redundancy using optimization techniques," IEEE Trans. on Robotics and Automation, 5(4), pp. 529-533.