ME/SE 740

Lecture 17

6-R Single Strand Kinematic Chains and the Inverse Kinematics Problem

From last lecture we see that for such kinematic chains we have:

$$T_6^0 = A_1 A_2 \cdots A_6$$

where (for the Elbow manipulator):

$$\begin{aligned}
A_k &= \begin{pmatrix} \cos\theta_k & -\sin\theta_k\cos\alpha_k & \sin\theta_k\sin\alpha_k & a_k\cos\theta_k \\ \sin\theta_k & \cos\theta_k\cos\alpha_k & -\cos\theta_k\sin\alpha_k & a_k\sin\theta_k \\ 0 & \sin\alpha_k & \cos\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
A_1 &= \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ \sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
A_i &= \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & 0 & a_i\sin\theta_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad i = 2, 3 \\
A_4 &= \begin{pmatrix} \cos\theta_4 & 0 & -\sin\theta_4 & a_4\cos\theta_4 \\ \sin\theta_4 & 0 & \cos\theta_4 & a_4\sin\theta_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
A_5 &= \begin{pmatrix} \cos\theta_5 & 0 & \sin\theta_5 & 0 \\ \sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
A_6 &= \begin{pmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Using the trigonometric identity:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

we can express the product $A_1 A_2 \cdots A_6$ as (where we denote $\cos(\theta_1+\theta_2+\theta_3)$ as c_{123}, etc.)

$$A_1 A_2 \cdots A_6 =$$
$$\begin{pmatrix} c_1 c_{234} c_5 c_6 - s_1 s_5 s_6 - c_1 s_{234} s_6 & -c_1 c_{234} c_5 s_6 + s_1 s_5 s_6 - c_1 s_{234} c_6 & c_1 c_{234} s_5 + s_1 c_5 & a_4 c_1 c_{234} + a_3 c_1 c_{23} + a_2 c_1 c_2 \\ s_1 c_{234} c_5 c_6 + c_1 s_5 c_6 - s_1 s_{234} s_6 & -s_1 c_{234} c_5 c_6 - s_1 s_{234} c_6 - c_1 s_5 s_6 & s_1 c_{234} s_5 - c_1 c_5 & a_4 s_1 c_{234} + a_3 s_1 c_{23} + a_2 s_1 c_2 \\ s_{234} c_5 c_6 + c_{234} s_6 & -s_{234} c_5 c_6 + c_{234} c_6 & s_{234} s_5 & a_4 s_{234} + a_3 s_{23} + a_2 s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Inverse Kinematics Problem

The problem can be simply stated as: Given a nonlinear relationship $f(\theta) = x$, solve for θ given x.

As a first case, consider the following inverse kinematics problem in the plane with a 2-link manipulator (see figure below) where we are given (x, y) (the coordinates of the end-effector) and seek to find the θ_1, θ_2 that correspond to it. In fact, we are interested in knowing how many solutions exist:

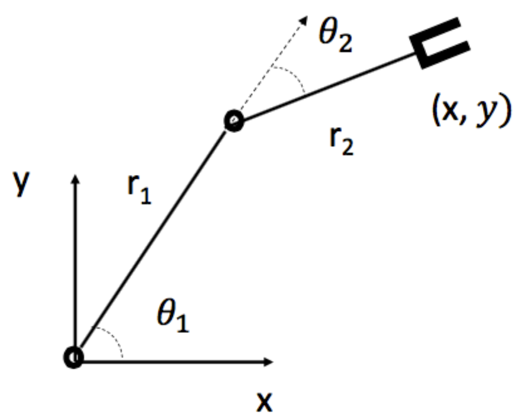

Figure 1: **Two link manipulator**

We have:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \cos\theta_1 + r_2 \cos(\theta_1+\theta_2) \\ r_1 \sin\theta_1 + r_2 \sin(\theta_1+\theta_2) \end{pmatrix}$$

Let:

$$x_1 = \cos\theta_1, \quad y_1 = \sin\theta_1, \quad x_2 = \cos(\theta_1+\theta_2), \quad y_2 = \sin(\theta_1+\theta_2)$$

This substitution transforms a system of two nonlinear equations in 2 unknowns into a system of 4 equations in 4 unknowns as can be seen below:

$$x = r_1x_1 + r_2x_2, \quad y = r_1y_1 + r_2y_2, \quad x_1^2 + y_1^2 = 1, \quad x_2^2 + y_2^2 = 1$$

Bezout's Theorem: Consider a family of n polynomial equations in n unknowns. An upper bound on the number of solutions (complex) (if it is finite) by the product of the degrees of the polynomials.

Elimination Theory:

$$1)\ \ x = r_1x_1 + r_2x_2, \quad 2)\ \ y = r_1y_1 + r_2y_2, \quad 3)\ \ x_1^2 + y_1^2 = 1, \quad 4)\ \ x_2^2 + y_2^2 = 1$$

Solve the 2^{nd} equation and write:

$$r_2^2y_2^2 = (y - r_1y_1)^2$$

Multiply the 4^{th} equation by r_2^2:

$$r_2^2x_2^2 + r_2^2y_2^2 = r_2^2$$

Use the last two equations to eliminate y_2:

$$r_2^2x_2^2 + (y - r_1y_1)^2 = r_2^2$$

Use this equation and equation 1) above to eliminate x_2:

$$r_2^2x_2^2 = (x - r_1x_1)^2, \quad \text{which yields} \quad (x - r_1x_1)^2 + (y - r_1y_1)^2 = r_2^2$$

Through this elimination process we end up with 2 equations and 2 unknowns:

$$\begin{aligned} (x - r_1x_1)^2 + (y - r_1y_1)^2 &= r_2^2 \\ x_1^2 + y_1^2 &= 1 \end{aligned}$$

Let us re-write these two equations and examine what they mean "geometrically:"

$$\begin{aligned} (x_1 - \frac{x}{r_1})^2 + (y_1 - \frac{y}{r_1})^2 &= \frac{r_2^2}{r_1^2} \\ x_1^2 + y_1^2 &= 1 \end{aligned}$$

One can see that the solution of these two equations represents the intersection of two circles (see figure below):

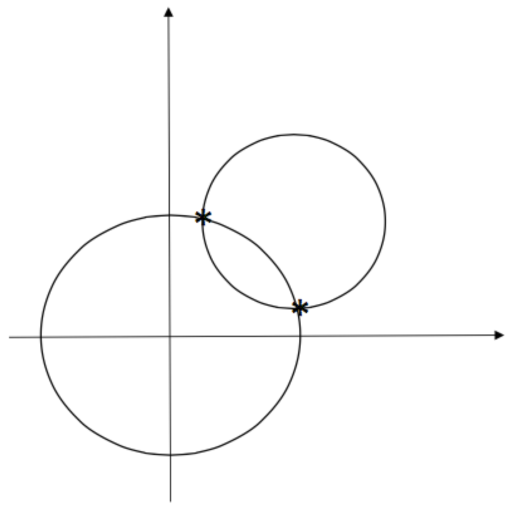

Figure 2: **Two Solutions**

These two solutions correspond to the two manipulator configurations "Elbow Up" and "Elbow Down:"

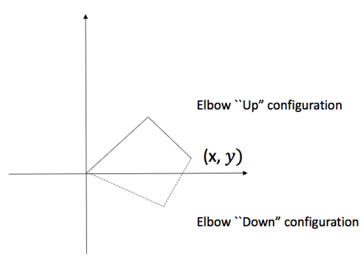

Figure 3: **Two Configurations**

It is unfortunate that the Bezout bound is not tight. Don Pieper's Thesis (1968) predicted $64,000$ solutions for general 6-dof mechnisms:

$$A_1(\theta_1)A_2(\theta_2)A_3(\theta_3)A_4(\theta_4)A_5(\theta_5)A_6(\theta_6) = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Make the same substitutions:

$$x_i = \cos\theta_i, \quad y_i = \sin\theta_i, \quad x_i^2 + y_i^2 = 1 \qquad \text{add constraints}$$

Compute the product of the degrees:

- 12 equations of degree 6 in x_i, y_i
- 6 equations of degree 2
- Product of the degrees: $6^{12}2^6 = 139,314,069,504$

Multiplicities of solutions can change: This is something that can usefully be understood using "Inverse Function Theorem" ideas (i.e., there exist locally unique solutions when J is invertible, $|J| \neq 0$):

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{pmatrix} = \begin{pmatrix} r_1 \cos\theta_1 + r_2 \cos(\theta_1 + \theta_2) \\ r_1 \sin\theta_1 + r_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

Then:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} \end{pmatrix} = \begin{pmatrix} -r_1 \sin\theta_1 - r_2 \sin(\theta_1 + \theta_2) & -r_2 \sin(\theta_1 + \theta_2) \\ r_1 \cos\theta_1 + r_2 \cos(\theta_1 + \theta_2) & r_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

That makes:

$$\begin{aligned} detJ &= r_1 r_2 (\sin(\theta_1 + \theta_2)\cos\theta_1 - \sin\theta_1 \cos(\theta_1 + \theta_2)) \\ &= r_1 r_2 \sin\theta_2 \end{aligned}$$

Kinematic singularity exists when $\theta_2 = 0$, or $\theta_2 = 180°$.

Note: Kinematic singularities are configurations where the Jacobian function looses rank:

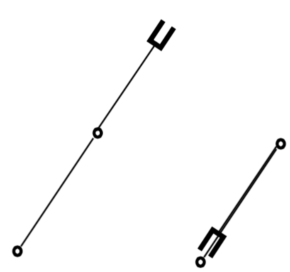

Figure 4: **Two Configurations**