

# ME/SE 740

## Lecture 16

### Homogeneous Transformations, $SE(3)$ Elements and D-H Parameters

Consider the coordinate frames associated with two consecutive links of some kinematic chain shown below. The  $k^{th}$  coordinate frame is determined axes  $x_k, y_k, z_k$  with its associated DH-Parameters:

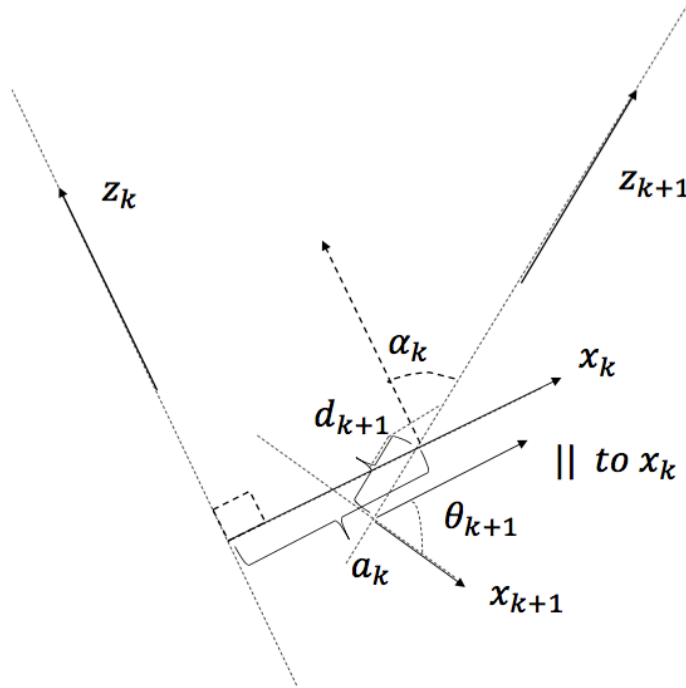


Figure 1: Link frames from two consecutive links

**Question:** How is a point referred to in the  $k^{th}$  frame described in the  $k + 1$  frame?

We answer this question by defining intermediate coordinate systems  $R, Q, P, S$  as follows:

- Frame R: It differs from frame  $k$  by a translation of  $a_k$  units along axis  $x_k$
- Frame Q: It differs from frame  $R$  by a rotation  $\alpha_k$  about axis  $x_k$
- Frame P: It differs from frame  $Q$  by a translation  $d_{k+1}$  along axis  $z_{k+1}$
- Frame S: It differs from frame  $P$  by a rotation  $\theta_{k+1}$  about axis  $z_{k+1}$

The matrices that represent the coordinate transformations are:

$$R = \begin{pmatrix} 1 & 0 & 0 & a_k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha_k & -\sin \alpha_k & 0 \\ 0 & \sin \alpha_k & \cos \alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{k+1} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \cos \theta_{k+1} & -\sin \theta_{k+1} & 0 & 0 \\ \sin \theta_{k+1} & \cos \theta_{k+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given a point with coordinates  $x_{k+1}, y_{k+1}, z_{k+1}$  in the  $k+1$  frame, the coordinates in the  $k^{\text{th}}$  frame are given by the formula:

$$\begin{pmatrix} x_k \\ y_k \\ z_k \\ 1 \end{pmatrix} = RQPS \begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ 1 \end{pmatrix}$$

Note::

$$RQ = \begin{pmatrix} 1 & 0 & 0 & a_k \\ 0 & \cos \alpha_k & -\sin \alpha_k & 0 \\ 0 & \sin \alpha_k & \cos \alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{Screw}(X_k, a_k, \alpha_k)$$

$$PS = \begin{pmatrix} \cos \theta_{k+1} & -\sin \theta_{k+1} & 0 & 0 \\ \sin \theta_{k+1} & \cos \theta_{k+1} & 0 & 0 \\ 0 & 0 & 1 & d_{k+1} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{Screw}(Z_{k+1}, d_{k+1}, \theta_{k+1})$$

$$\vec{P}_k = \text{Screw}(X_k, a_k, \alpha_k) \text{Screw}(Z_{k+1}, d_{k+1}, \theta_{k+1}) \vec{P}_{k+1}$$

Once the joint frames have been defined and the corresponding parameters found the joint frame transformation can be multiplied together to find a single transformation that relates frame  $\{N\}$  (at the tip frame) to frame  $\{0\}$  (at the base frame):

$$T_N^0 = T_1^0 T_2^1 \dots T_N^{N-1}$$

Recall from our convention that  $\alpha_0 = 0, a_0 = 0$  hence we may also write that:

$$T_N^0 = A_1 A_2 \cdots A_N$$

where:

$$\begin{aligned} A_k &= \text{Screw}(Z_k, d_k, \theta_k) \text{Screw}(X_k, a_k, \alpha_k) \\ &= \begin{pmatrix} \cos \theta_k & -\sin \theta_k & 0 & 0 \\ \sin \theta_k & \cos \theta_k & 0 & 0 \\ 0 & 0 & 1 & d_k \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_k \\ 0 & \cos \alpha_k & -\sin \alpha_k & 0 \\ 0 & \sin \alpha_k & \cos \alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta_k & -\sin \theta_k \cos \alpha_k & \sin \theta_k \sin \alpha_k & a_k \cos \theta_k \\ \sin \theta_k & \cos \theta_k \cos \alpha_k & -\cos \theta_k \sin \alpha_k & a_k \sin \theta_k \\ 0 & \sin \alpha_k & \cos \alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$