ME/SE 740

Lecture 10

Lie Groups I

Review

- Geometric Relationships in the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$
- Group Theory
- Matrix exponentials

Definition: A subgroup of the group of all $n \times n$ invertible matrices is called a **Lie Group** (matrix Lie Group) if it is also a closed sub-manifold (you can do calculus on it).

Example: The "winding line" that is dense on the 2-torus is not a Lie Group.

Definition: A vector space V (over $\mathbb{R}$) is a **Lie Algebra** if in addition to the vector space structure there is defined a binary operation:

$$[\cdot,\cdot]: \quad V \times V \quad \longrightarrow \quad V \quad \text{called "Lie Bracket"}$$

satisfying the following properties:

$$\begin{array}{rll}
i) & \text{bilinearity property} & [a_1 v_1 + a_2 v_2, w] \;=\; a_1[v_1, w] + a_2[v_2, w] \quad \text{for all} \quad v_1, v_2, w \in V, \quad a_1, a_2, \in \mathbb{R} \\
ii) & \text{skew symmetry} & [v, w] = -[w, v] \quad \text{for all} \quad v, w \in V \\
iii) & \text{Jacobi Identity} & [v,[w,z]] + [w,[z,v]] + [z,[v,w]] = 0 \quad \text{for all} \quad v, w, z \in V
\end{array}$$

Three 3×3 Examples

1. Vector cross product in $\mathbb{R}^3$, $\quad v, w, \in \mathbb{R}^3, \quad [v, w] \;=\; v \times w \quad \in \mathbb{R}^3$
2. $so(3)$, vector space of 3×3 skew-symmetric matrices, (with $[A, B] = AB - BA$)
3. $s\ell(2)$, set of all 2×2 matrices with trace $= 0$, (with $[A, B] = AB - BA$)

Given a matrix Lie Group G, we wish to study the tangent space at the identity. Let $S(t)$ be a curve in G such that $S(0) = I$, $\;S'(0) = A$ (element in the tangent space). Let $R \in G$. Then $T(t) = RS(t)R^{-1}$ and $T(0) = I$. Hence $T'(0) = RS'(0)R^{-1} = RAR^{-1}$ is in the tangent space at the identity.

Proposition: For any $R \in G$, if A is in the tangent space at the identity, $T_I G$, then $\underbrace{RAR^{-1}}_{conjugation}$, is also in $T_I G$.

Proposition: Let $R(t)$ be a curve in G such that $R(0) = I$, $\;R'(0) = B$. Let A be an element of $T_I G$. Then:

$$\begin{array}{ll} i) & R(t)AR(t)^{-1} \text{is a curve in} T_I G \\ ii) & \frac{d}{dt}|_{t=0} R(t)AR(t)^{-1} = BA - AB \end{array}$$

proof: statement i) repeats the previous Proposition. To show statement ii) we must evaluate $\frac{d}{dt}[R^{-1}(t)]$.

Note:

$$R(t)R^{-1}(t) = I, \quad \text{hence} \quad R'(t)R(t)^{-1} + R(t)\frac{d(R^{-1}(t))}{dt} = 0$$

$$\Longrightarrow \quad R'(0)R(0)^{-1} + R(0)\frac{d}{dt}|_{t=0}(R^{-1}(t)) = 0 = B \cdot I + I \cdot \frac{d}{dt}|_{t=0}(R^{-1}(t))$$

$$\Longrightarrow \quad \frac{d}{dt}|_{t=0}(R^{-1}(t)) = -R'(0) = -B$$

$$\Longrightarrow \quad \frac{d}{dt}|_{t=0}(R(t)AR^{-1}(t)) = R'(0)A + A\frac{d}{dt}|_{t=0}(R^{-1}(t)) = BA - AB$$

The expression $BA - AB$ is known as the matrix Lie Bracket, $[B, A] = BA - AB$.

Proposition: Given a matrix Lie Group G, the tangent space at the identity $T_I G$ is a Lie Algebra with respect to this Lie Bracket.

Note: Velocities “live” in some transformed space of Lie Algebras.

Example 1: If J is any nonsingular $n \times n$ matrix, the set of all $n \times n$ nonsingular matrices M such that $M^T JM = J$ is a group (with respect to ordinary matrix multiplication).

proof: We will show that i) it is closed under matrix multiplication and ii) it is closed under the operation of taking inverses:

i)

$$\left.\begin{array}{l} M_1^T JM_1 = J \\ M_2^T JM_2 = J \end{array}\right\} \quad \Longrightarrow (M_1M_2)^T J(M_1M_2) = M_2^T \underbrace{M_1^T JM_1}_{J} M_2 = J$$

ii)

$$\begin{array}{rcl} M^T JM & = & J, \quad \text{does this imply} \ (M^{-1})^T JM^{-1} = J? \\ (M^T JM)M^{-1} & = & JM^{-1} \quad \Longrightarrow \quad M^T J = JM^{-1} \quad \Longrightarrow \quad J = (M^T)^{-1}JM^{-1} = (M^{-1})^T JM^{-1} \end{array}$$

<u>SPECIAL CASE</u>: $J = I, \quad \Longrightarrow \quad \mathcal{G} = O(n), \quad n \times n$ orthogonal matrices.

Example 2: With $n = 2m$ and

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{G} = Sp(2m), \quad \text{the symplectic group}$$

Example 3:

$$J = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{G} \quad \text{the Lorentz group}$$

Let us consider the tangent space at the identity $T_I G$. Let $R(t)$ be a curve in $\mathcal{G}$. Then $R(t)^T J R(t) = J$ (The invariance property at the group level).

Assume $R(0) = I$, and write $R'(0) = A$. Differentiating both sides of the group invariance property at $t = 0$ we obtain:

$$R'(0)^T J + JR'(0) = A^T J + JA = 0$$

This is the corresponding invariance property for the Lie Algebra.

Special Case: $J = I$, $A^T + A = 0$.

Let α be a set of $n \times n$ matrices that is closed with respect to vector space operations and also with respect to the matrix Lie bracket $A, B, \in \alpha \implies [A, B] = AB - BA \in \alpha$. In other words α is a matrix Lie algebra. If α is such a Lie algebra, the set of all finite products:

$$e^{A_1} \cdot e^{A_2} \cdots e^{A_k}, \quad k \in \mathbb{Z}^+, \ A_j \in \alpha \ , t_j \in \mathbb{R}$$

is the corresponding <u>matrix Lie group</u>.

Example 1: If α is the Lie algebra of all $n \times n$ matrices the corresponding Lie group is the group of $n \times n$ invertible matrices.

Example 2: Let $G = SO(3)$, (set of 3×3 orthogonal matrices with determinant equal to 1), and $\alpha = so(3)$ (set of 3×3 skew symmetric matrices).

Consider the basis for $so(3)$

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

The Lie bracket of two of them gives the third (possible with a "-" sign) as:

$$\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The group $SO(3)$ can be thought of as all products ($t_i' s \in \mathbb{R}$):

$$e^{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} t_1} e^{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} t_2} e^{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} t_3}$$

One can show that:

$$e^{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} t_1} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} t_1 + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} t_1^2 + \frac{1}{3!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} t_1^3 + \frac{1}{4!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} t_1^4 + \cdots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t_1 & -\sin t_1 \\ 0 & \sin t_1 & \cos t_1 \end{pmatrix}$$